An Introduction to Hypernetworks

Lesson 3. Hypernetworks

An Étoile Course in association with the

4th Ph.D. Summer School - Conference on

Mathematical Modeling of Complex Systems

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1 Hypersimplices

A *hypersimplex* is a simplex that carries its defining relation explicitly following the vertices, *e.g.* the blocks b_1, b_2 and b_3 in Fig. 1 are combined by the relation R to create the hypersimplex $\sigma = \langle b_1, b_2, b_3\,; R \rangle$, where the “arch” σ exists at a higher more aggregate level than its parts. To emphasise their relational nature, hypersimplices may also be called *relational hypersimplices*.

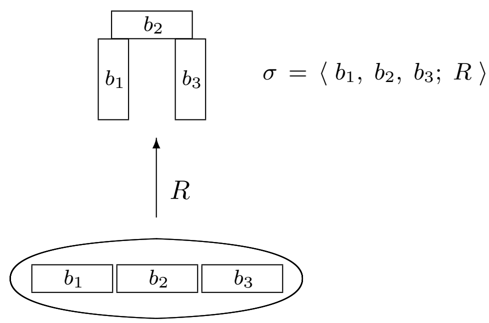


Figure 1: Parts combined into a whole by R forming a *relational hypersimplex*

Vertex Order in Relational Hypersimplices

In algebraic topology the vertex order in simplices is crucial. For example, swapping a pair of vertices gives something different, $\langle ...v_i...v_j... \rangle \neq \langle ...v_j...v_i... \rangle$. Vertex order is also necessary in relational hypersimplices, but it is not sufficient to distinguish different structures formed from the same vertices.

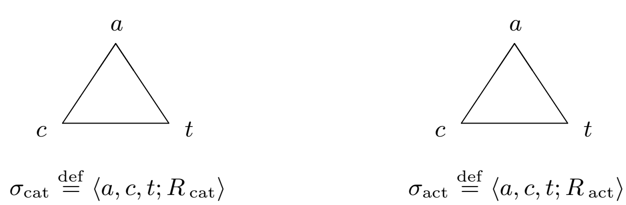


Figure 2: The relational simplices $\langle a, c, t; R_{\mathrm{cat}} \rangle$ and $\langle a, c, t; R_{\mathrm{act}} \rangle$

The words *act* and *cat* are formed by 3-ary relations on the letters a, c and t. They can be discriminated by ordering the vertices appropriately, since $\langle a, c, t \rangle \neq \langle c, a, t \rangle$. However, although vertex ordering can usefully discriminate simplices such as $\langle a, c, t \rangle$ and $\langle c, a, t \rangle$, it is not sufficient to differentiate all n-ary relational structure.

For example, Fig. 3 shows the three objects $\Diamond$, $\heartsuit$, and $\Box$ related in twelve ways. There are only six ways of ordering these objects as vertices, namely $\langle \Diamond, \heartsuit, \Box \rangle$, $\langle \Diamond, \Box, \heartsuit \rangle$, $\langle \heartsuit, \Diamond, \Box \rangle$, $\langle \heartsuit, \Box, \Diamond \rangle$, $\langle \Box, \Diamond, \heartsuit \rangle$, and $\langle \Box, \heartsuit, \Diamond \rangle$, and this is not enough to distinguish the twelve different relationships that define the configurations.

$\Box\heartsuit$	$\Box\Diamond$	$\heartsuit\Box$	$\heartsuit\Diamond$	$\Diamond\Box$	$\Diamond\heartsuit$	$\Diamond$	$\heartsuit$	$\Diamond$	$\Box$	$\heartsuit$	$\Box$
$\Diamond$	$\heartsuit$	$\Diamond$	$\Box$	$\heartsuit$	$\Box$	$\Box\heartsuit$	$\Box\Diamond$	$\heartsuit\Box$	$\heartsuit\Diamond$	$\Diamond\Box$	$\Diamond\heartsuit$
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}

Figure 3: Twelve configurations of the elements $\{\Diamond, \heartsuit, \Box\}$

To remove ambiguity it is necessary to make explicit the n-ary relation that defines any particular structure. For example, c_1 can be written as $\langle \Diamond, \heartsuit, \Box ; R_1 \rangle$ which can be discriminated from $c_7 = \langle \Diamond, \heartsuit, \Box ; R_7 \rangle$ as shown below:

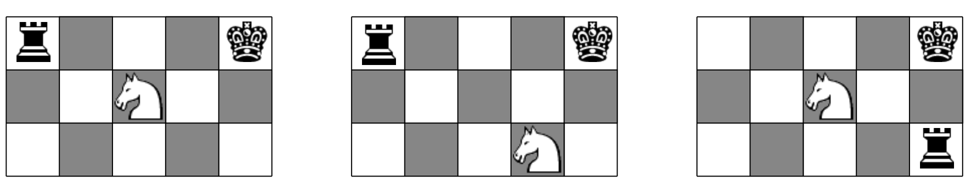
$$\langle \Diamond, \heartsuit, \Box \,; R_1 \rangle \;=\; \begin{array}{c} \Box\heartsuit \\ \Diamond \end{array} \;\neq\; \begin{array}{c} \Diamond \\ \Box\heartsuit \end{array} \;=\; \langle \Diamond, \heartsuit, \Box \,; R_7 \rangle$$

Hypernetwork theory extends conventional graphs, networks, hypergraphs and simplicial complexes to make explicit the n-ary relations. Relational hypersimplices allow structures to be discriminated, even when they have the same constituent parts, $\langle v_1, ..., v_n; R \rangle \neq \langle v_1, ..., v_n; R' \rangle$.

2 Examples

Example: The Knight Fork

Figure 4 shows three configurations of chess pieces. The configuration on the left, $\langle$rook, knight, king; $R_1\rangle$, is called a *knight fork* because the white knight threatens the black rook as it puts the black king in check. Unless black has a piece that can take it, the white knight can take the black rook because black must move the king out of check. The configuration in the centre, (b) $\langle$rook, knight, king; $R_2\rangle$, is not a knight fork, even through the knight puts the king in check. The configuration on the right is another knight fork, but it is clearly different to that on the left. Thus, the same three pieces are assembled by three different relations, R_1, R_2 and R_3 to form three different structures.



(a) $\langle$rook, knight, king; $R_1\rangle$ is a knight fork

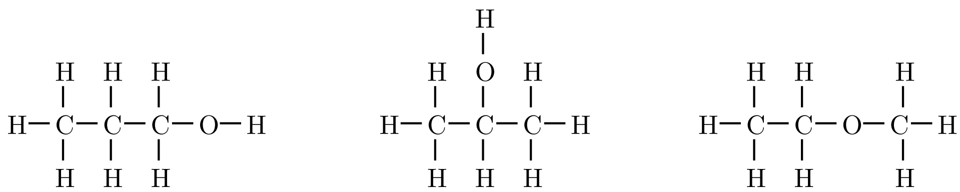
(b) $\langle$rook, knight, king; $R_2\rangle$ is not a knight fork

(c) $\langle$rook, knight, king; $R_3\rangle$ is a knight fork

Figure 4: Knight fork structures in chess

Example: Chemical Isomers

Chemical molecules are assemblies of atoms. For example propanol assembles three carbon atoms with eight hydrogen atoms and an oxygen atom, written as C_3H_8O or C_3H_7OH.



(a) *n*-propyl alcohol (b) isopropyl alcohol (c) methyl-ethyl-ether

Figure 5: Chemical isomers as relational simplices

Figure 5 shows the atoms of propanol arranged in a variety of ways. The first two show the isomers *n*-propyl alcohol and isopropyl alcohol. The oxygen atom is attached to an end carbon in the first isomer and to the centre carbon in the second, but the C-O-H hydroxyl group substructure is common to both. The rightmost isomer of C_3H_8O, methoxyethane, has the oxygen atom connected to two carbon atoms and there is no C-O-H substructure. This makes it an ether, methyl-ethyl-ether, rather than an alcohol. Thus the hypersimplices of the isomers have the same vertices, but the assembly relations are different. *n*-propyl alcohol and isopropyl alcohol share the hydroxyl group substructure C-O-H and are similar, but methyl-ethyl-ether does not and has different properties. Thus

$$\begin{aligned} &\langle\, \mathrm{C, C, C, H, H, H, H, H, H, H, H, O}\,;\, R_{\,n-\mathrm{propylalcohol}}\rangle \quad \neq \\ &\langle\, \mathrm{C, C, C, H, H, H, H, H, H, H, H, O}\,;\, R_{\,\mathrm{isopropylalcohol}}\rangle \quad \neq \\ &\langle\, \mathrm{C, C, C, H, H, H, H, H, H, H, H, O}\,;\, R_{\,\mathrm{methyl-ethyl-ether}}\rangle \end{aligned}$$

Example: The Perfect Gin and Tonic

Figure 6: The perfect gin and tonic, $\langle$gin, tonic, ice, lemon; $R_{perfect}\,\rangle$

A 'gin and tonic' cocktail is made from gin, tonic, ice and a lemon. For me the way to make a 'perfect' gin and tonic is the 4-ary relation, $R_{perfect}$, defined as "Put the ice in a glass. Add a slice of lemon. Pour gin over the ice. Add tonic, and stir." This gives the hypersimplex $\langle$gin, tonic, ice, lemon; $R_{perfect}\,\rangle$ as illustrated in Figure 6.

Example: IQ Questions

Figure 7 is based on an a question in an IQ test. What is your answer?

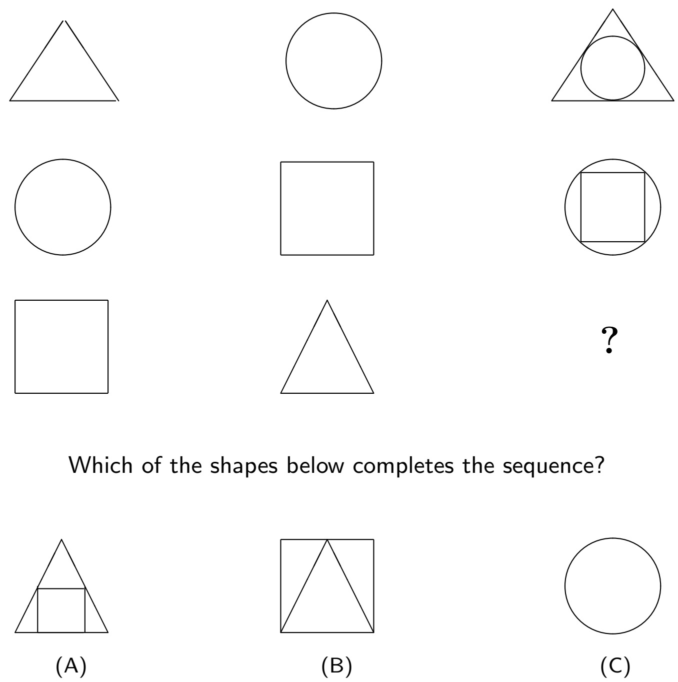


Figure 7: An IQ test question

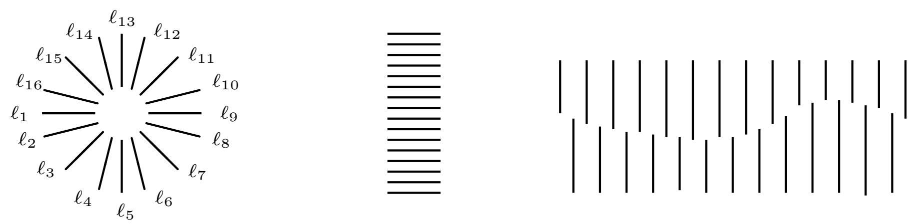
To answer the question, one line of reasoning goes as follows: on the top row the circle (second object) is inside the triangle (first object); on the second row the square (first object) is inside the circle (second object); to follow the pattern on the last row the second object (a triangle) should be inside the first object (a square) and the answer is (B). Thus the structure behind this question is the relational hypersimplex $\langle x_1, x_2; R_{x_2 \text{ is inside x}_1} \rangle$. Although one might be tempted to answer (A), this is not the right pattern because its relation is $R_{x_1 \text{ is inside x}_2}$ rather than the correct relation $R_{x_2 \text{ is inside x}_1}$.

Another question in this IQ test asked, "A forest is to a tree as a tree is to a ?", giving the options orchard, plant, jungle and leaf. The relation $R_{x_2 \text{ is part of } x_1}$ applies to forest and tree, which suggests the answer is given by relational hypersmplex simplex $\langle$ tree, leaf; $R_{x_2 \text{ is part of } x_1} \rangle$. If so the answer is "leaf".

Another question asked, "Car is to road as train is to", giving the options surface, locomotive, rails and wheels. Here the relational hypersimplex simplex is $\langle x_1, x_2; R_{\text{travels on}} \rangle$ and the likely answer is "rails".

Example: The Sun Illusion and Virtual Contours

Figure 8(a) shows the set of lines $\ell_1, ..., \ell_{16}$ arranged in a circle by the relation R_1. The resulting structure $\langle \ell_1, ..., \ell_{16}; R_1 \rangle$ has the emergent property that most people see a clear white disk at the centre of the lines, the so-called *sun illusion*. Figure 8(b) shows the same set of lines assembled under a different relation, R_2. Now there is no disk but a rectangle shape emerges. Figure 8(c) shows a twenty eight lines assembled in such a way that a so-called *virtual contour* emerges.



(a) The sun illusion $\sigma_1 = \langle \ell_1, ..., \ell_{16}; R_1 \rangle$

(b) the rectangle illusion $\sigma_2 = \langle \ell_1, ..., \ell_{16}; R_2 \rangle$

(c) the virtual contour illusion $\sigma_3 = \langle \ell_1, ..., \ell_{28}; R_3 \rangle$

Figure 8: Emergent features in line assemblies

3 Hypernetworks

Any set of hypersimplices forms a *hypernetwork*. Like simplices, hypersimplices have a *geometric realisation* as polyhedra. Hypernetworks have the same connectivity properties as simplicial complexes, including the notions of q-connectivity, eccentricity, and Galois pairs.

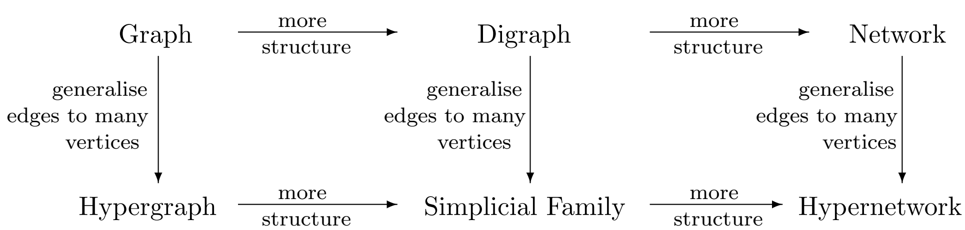


Figure 9: Hypernetworks generalise all the common network structures

Figure 9 shows how the common relational structures form a unified whole. On the top line, relations between pairs of things are given more structure. Vertically there is generalisation from binary relations between pairs of things to n-relations between any number of things. On the bottom line, hypergraphs edges become simplices when the vertices are ordered, and simplices become hypersimplices when n-ary relations are made explicit.

Hypernetworks do not compete with hypergraphs or networks – they naturally generalise both. Hypernetworks, sets of hypersimplices, provide the last piece in the relational jigsaw.